On the notion of "ground state" for the nonlinear Schrödinger equation on metric graphs

Séminaire d'analyse appliquée A^3 - LAMFA

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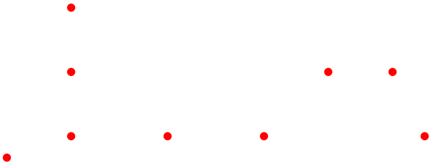
Monday 16 January 2023

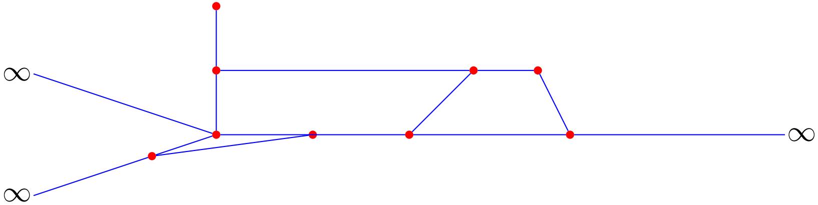
What is a metric graph?

A metric graph is made of vertices



What is a metric graph?

A metric graph is made of vertices and of edges joining the vertices or going to infinity.



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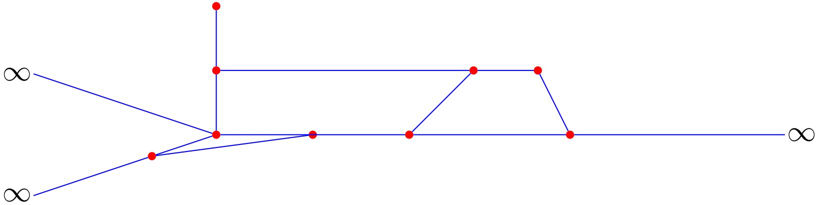

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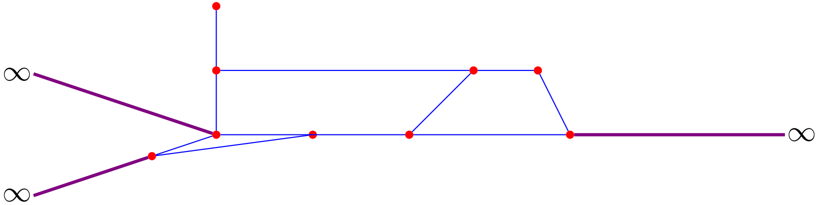

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- the edges going to infinity are halflines and have *infinite length*.

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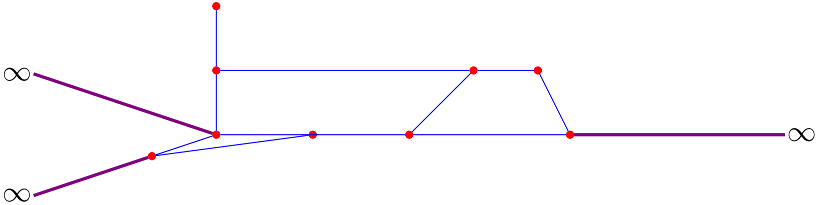
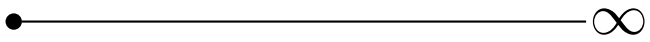


- *metric* graphs: the length of edges are important.
- the edges going to infinity are halflines and have *infinite length*.
- a metric graph is *compact* if and only if it has a finite number of edges of finite length.

Constructions based on halflines



The halfline

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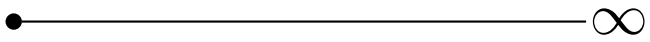
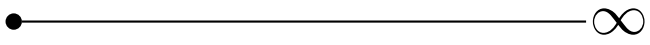


The halfline

The line

Constructions based on halflines



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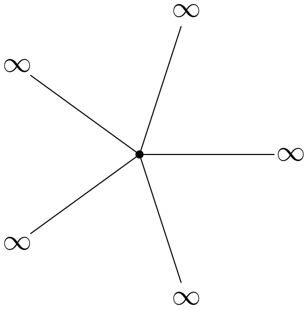

The 5-star graph

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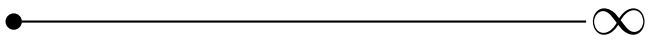

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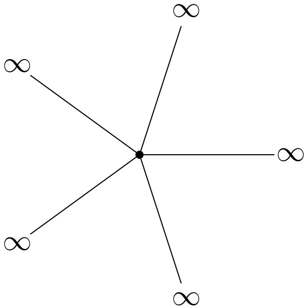

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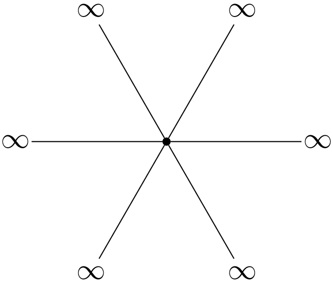
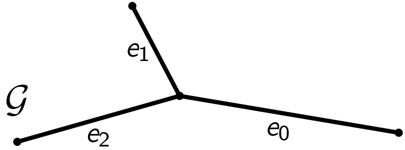


The 6-star graph

Functions defined on metric graphs



A metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3)

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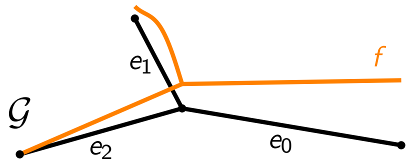

A metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3),
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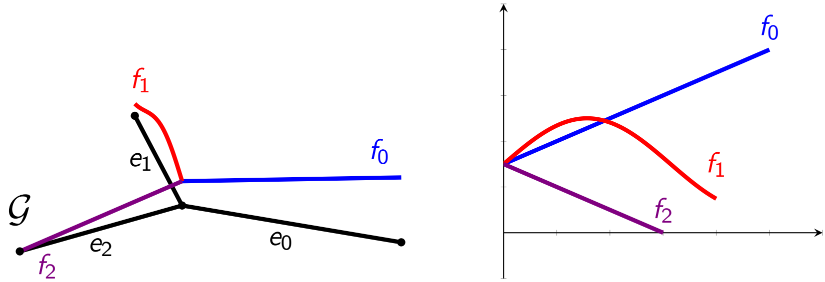

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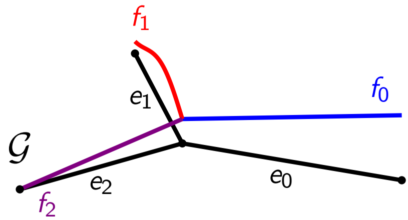

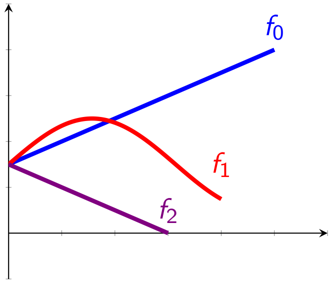

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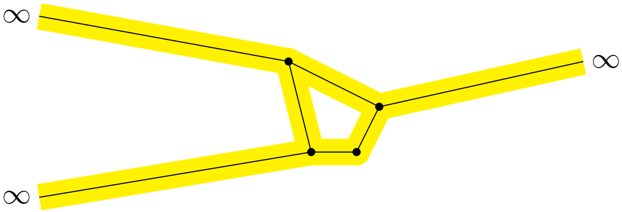
$$\int_{\mathcal{G}} f \, \mathrm{d}x \overset{\text{def}}{=} \int_0^5 f_0(x) \, \mathrm{d}x + \int_0^4 f_1(x) \, \mathrm{d}x + \int_0^3 f_2(x) \, \mathrm{d}x$$

Why studying metric graphs?
Physical motivations

Modeling structures where *only one spatial direction is important.*



A « fat graph » and the underlying metric graph

The differential system

Given constants $p > 2$ and $\lambda > 0$, we are interested in solutions $u \in L^2(\mathcal{G})$ of the differential system

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where the symbol $e \succ \mathrm{v}$ means that the sum ranges over all edges of vertex v and where $\frac{\mathrm{d}u}{\mathrm{d}x_e}(\mathrm{v})$ is the outgoing derivative of u at v (*Kirchhoff's condition*).

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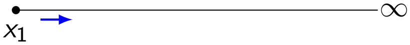
We denote by $\mathcal{S}_\lambda(\mathcal{G})$ the set of solutions of the differential system.

Kirchoff's condition: degree one nodes



$$\lim_{t \xrightarrow[t>0]{} 0} \frac{u(x_1+t)-u(x_1)}{t} = 0$$

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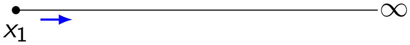
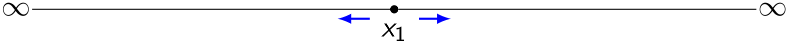


$$\lim_{\substack{t \longrightarrow 0 \\ t>0}} \frac{u(x_1+t)-u(x_1)}{t} = 0$$

In other words, the derivative of u at x_1 vanishes: this is the usual Neumann condition.

Kirchoff's condition: degree two nodes



$$\left(\lim_{t \xrightarrow[t>0]{} 0} \frac{u(x_1+t)-u(x_1)}{t}\right) + \left(\lim_{t \xrightarrow[t>0]{} 0} \frac{u(x_1-t)-u(x_1)}{t}\right) = 0$$

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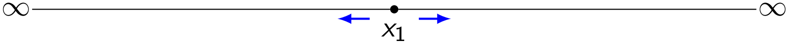

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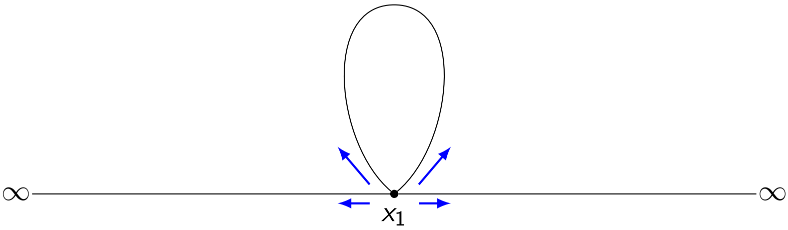
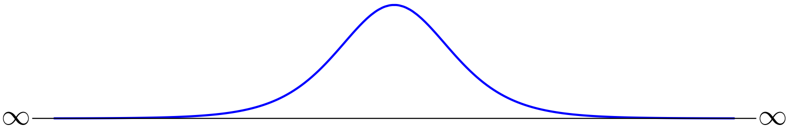
In other words, the left and right derivatives of u are equal, which simply means that u is differentiable at x_1. This explains why usually we do not put degree two nodes.

Kirchoff's condition in general: outgoing derivatives



$$\sum_{e \succ \mathrm{v}} \frac{\mathrm{d}u}{\mathrm{d}x_e}(\mathrm{v}) = 0$$

The real line: $\mathcal{G} = \mathbb{R}$



$$\mathcal{S}_\lambda(\mathbb{R}) = \left\{ \pm\varphi_\lambda(x+a) \;\middle|\; a \in \mathbb{R} \right\}$$

where the *soliton* φ_λ is the unique strictly positive and even solution to

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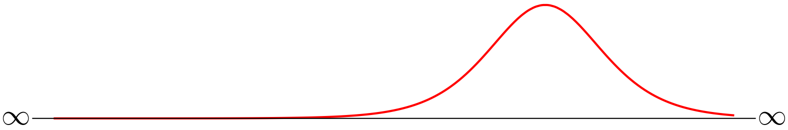

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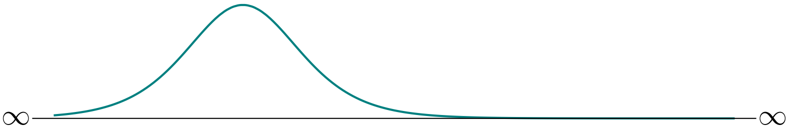

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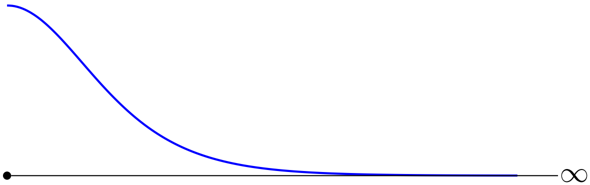
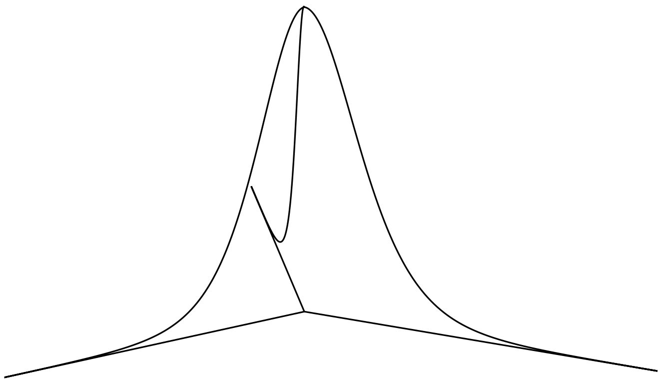
The halfline: $\mathcal{G} = \mathbb{R}^+ = [0, +\infty[$



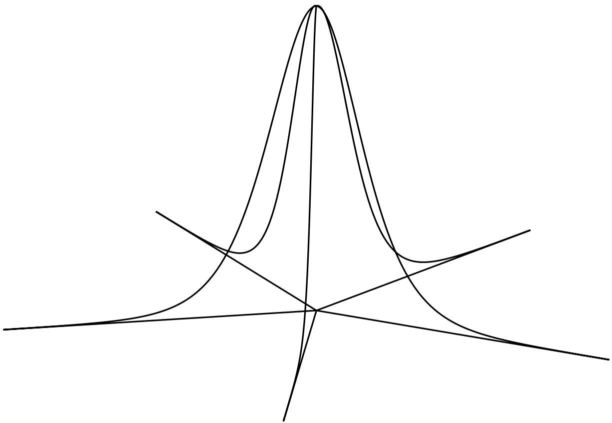
$$\mathcal{S}_\lambda(\mathbb{R}^+) = \left\{ \pm\varphi_\lambda(x)_{|\mathbb{R}^+} \right\}$$

Solutions are *half-solitons*: no more translations!

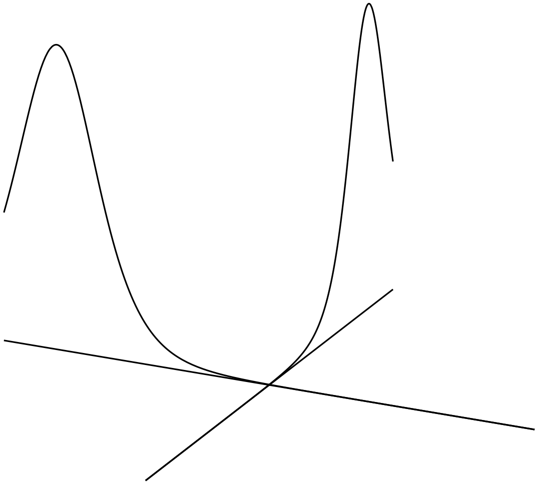
The positive solution on the 3-star graph



The positive solution on the 5-star graph



A continuous family of solutions on the 4-star graph



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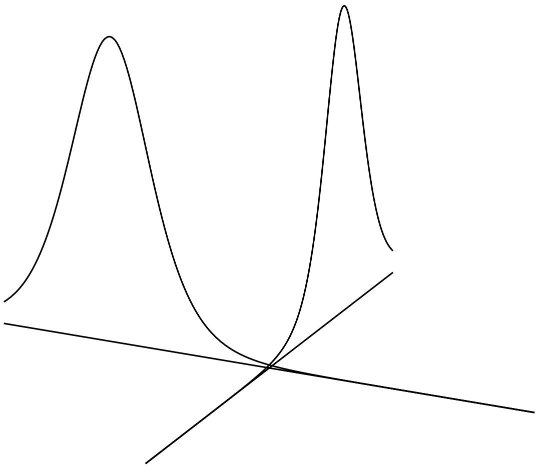

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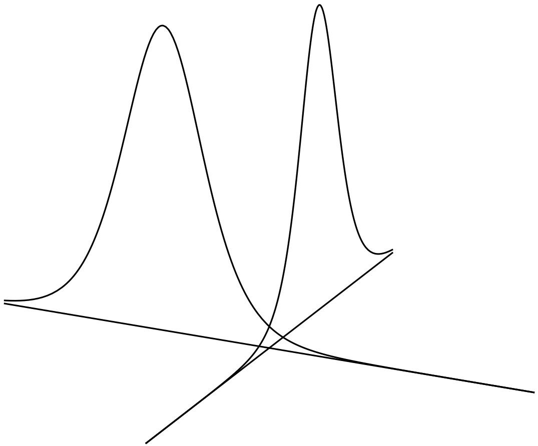

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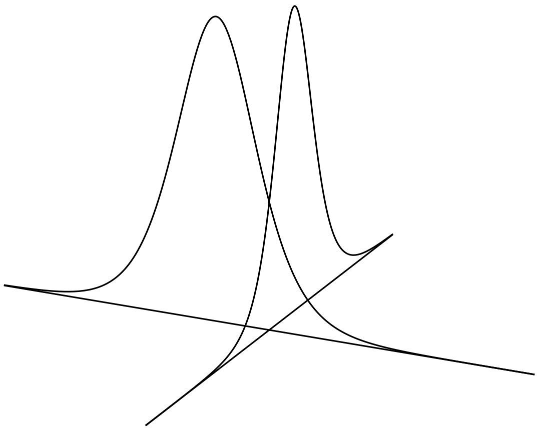

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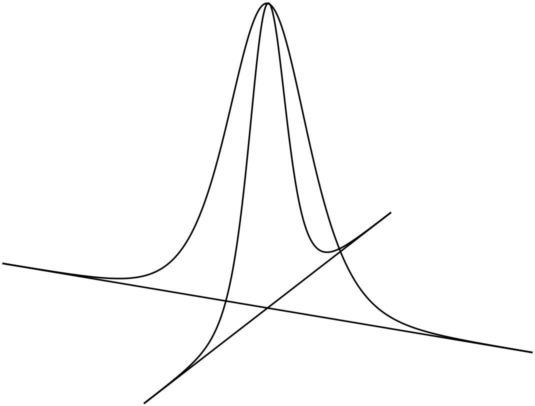

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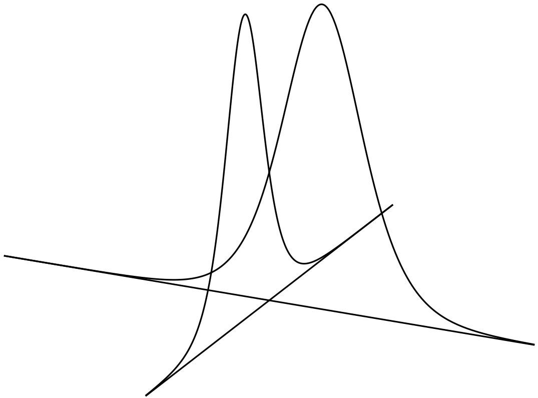

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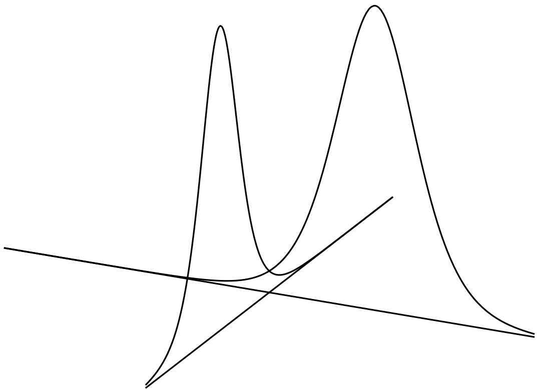

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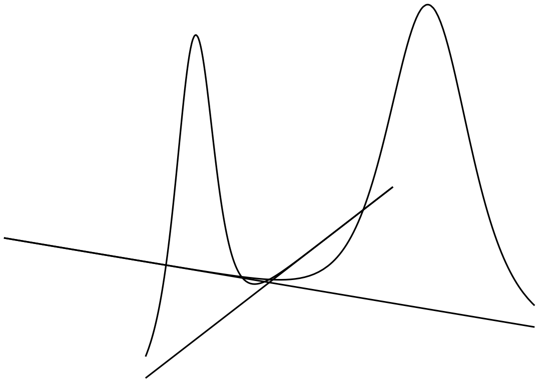

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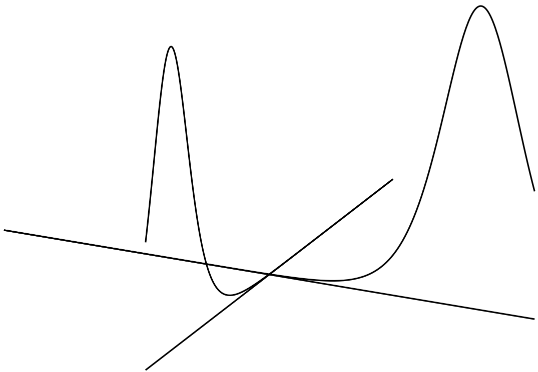

Variational formulation

We work on the Sobolev space

$$H^1(\mathcal{G}) := \Big\{ u : \mathcal{G} \to \mathbb{R} \mid u \text{ is continuous}, u, u' \in L^2(\mathcal{G}) \Big\}.$$

Variational formulation

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$$H^1(\mathcal{G}) := \left\{ u : \mathcal{G} \to \mathbb{R} \mid u \text{ is continuous}, u, u' \in L^2(\mathcal{G}) \right\}.$$

Solutions of (NLS) correspond to critical points of the *action functional*

$$J_\lambda(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{1}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})}.$$

The Euler-Lagrange equation associated to J_λ

The differential of $J_\lambda : H^1(\mathcal{G}) \to \mathbb{R}$ is given by

$$J_\lambda'(u)[v] = \int_{\mathcal{G}} u'(x)v'(x)\,\mathrm{d}x + \lambda \int_{\mathcal{G}} u(x)v(x)\,\mathrm{d}x - \int_{\mathcal{G}} |u(x)|^{p-2}u(x)v(x)\,\mathrm{d}x$$

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If φ has compact support in the interior of an edge $e = \mathrm{AB}$, we have

$$0 = J_\lambda'(u)[\varphi]$$

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&= \int_e u'(x)\varphi'(x)\,\mathrm{d}x + \lambda \int_e u(x)\varphi(x)\,\mathrm{d}x - \int_e |u(x)|^{p-2}u(x)\varphi(x)\,\mathrm{d}x \\
&= \frac{\mathrm{d}u}{\mathrm{d}x_e}(\mathrm{B})\underbrace{\varphi(\mathrm{B})}_{=0} - \frac{\mathrm{d}u}{\mathrm{d}x_e}(\mathrm{A})\underbrace{\varphi(\mathrm{A})}_{=0} \\
&\quad + \int_e (-u''(x) + \lambda u(x) - |u(x)|^{p-2}u(x))\varphi(x)\,\mathrm{d}x
\end{aligned}$$

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&= \int_e u'(x)\varphi'(x)\,\mathrm{d}x + \lambda \int_e u(x)\varphi(x)\,\mathrm{d}x - \int_e |u(x)|^{p-2}u(x)\varphi(x)\,\mathrm{d}x \\
&= \frac{\mathrm{d}u}{\mathrm{d}x_e}(\mathrm{B})\underbrace{\varphi(\mathrm{B})}_{=0} - \frac{\mathrm{d}u}{\mathrm{d}x_e}(\mathrm{A})\underbrace{\varphi(\mathrm{A})}_{=0} \\
&\quad + \int_e (-u''(x) + \lambda u(x) - |u(x)|^{p-2}u(x))\varphi(x)\,\mathrm{d}x
\end{aligned}$$

so that $u'' + |u|^{p-2}u = \lambda u$ on edges of $\mathcal{G}$.

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so that $\sum_{1\leq i\leq D} \frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(\mathrm{A}_i) = 0$, which is Kirchhoff's condition.

The Nehari manifold

The functional J_λ is not bounded from below on $H^1(\mathcal{G})$, since if $u \neq 0$ then

$$J_\lambda(tu) = \frac{t^2}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{t^2}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{t^p}{p}\|u\|^p_{L^p(\mathcal{G})} \xrightarrow[t\to\infty]{} -\infty.$$

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A common strategy is to introduce the *Nehari manifold* $\mathcal{N}_\lambda(\mathcal{G})$, defined by

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If $u \in \mathcal{N}_\lambda(\mathcal{G})$, then

$$J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^p_{L^p(\mathcal{G})}.$$

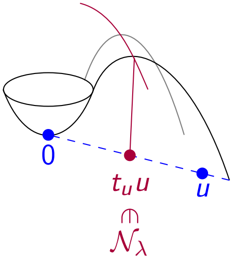
In particular, J_λ is bounded from below on $\mathcal{N}_\lambda(\mathcal{G})$.

The Nehari manifold

Geometry[1]

One can show that for every $u \in H^1(\mathcal{G}) \setminus \{0\}$, there exists a unique $t_u > 0$ so that $t_u u \in \mathcal{N}_\lambda(\mathcal{G})$, characterized by

$$J_\lambda(t_u u) = \max_{t>0} J_\lambda(tu).$$



[1] Thanks to C. Troestler for the picture!

Two energy levels

- « Ground state » energy level:

$$c_\lambda(\mathcal{G}) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u)$$

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- *Minimal action solution*: solution $u \in \mathcal{S}_\lambda(\mathcal{G})$ of the differential system (NLS) of level $\sigma_\lambda(\mathcal{G})$.

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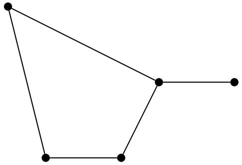
Theorem (De Coster, Dovetta, G., Serra (to appear))

For every $p > 2$, every $\lambda > 0$, and every choice of alternative between A1, A2, B1, B2, there exists a metric graph $\mathcal{G}$ where this alternative occurs.

Case A1

$c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G})$ and both infima are attained



Compact graphs

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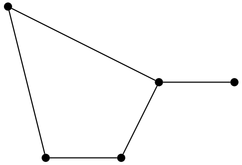

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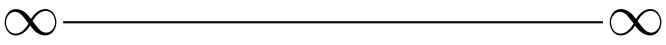

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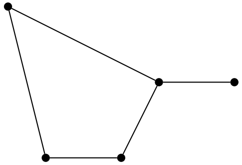

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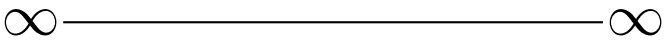

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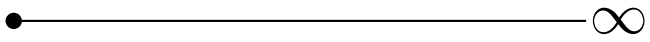

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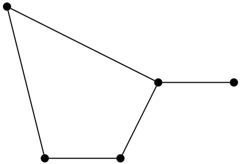

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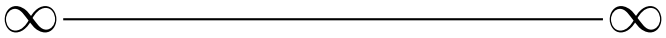

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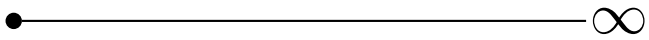

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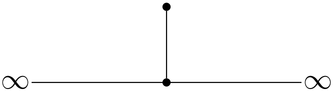

The line with one pendant

A very useful tool: cutting solitons on halflines

Proposition

Assume that $\mathcal{G}$ has at least one halfline. Then,

$$c_\lambda(\mathcal{G}) \leq s_\lambda := J_\lambda(\varphi_\lambda)$$

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Proof.

□

Case A1

$c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G})$ and both infima are attained

Theorem (Adami, Serra, Tilli 2014)

Let $\mathcal{G}$ be a metric graph with finitely many edges, including at least one halfline. Assume that

$$c_\lambda(\mathcal{G}) < s_\lambda.$$

Then $c_\lambda(\mathcal{G})$ is attained, which means that there exists a ground state, so we are in case A1: $c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G})$, both attained.

Case B1

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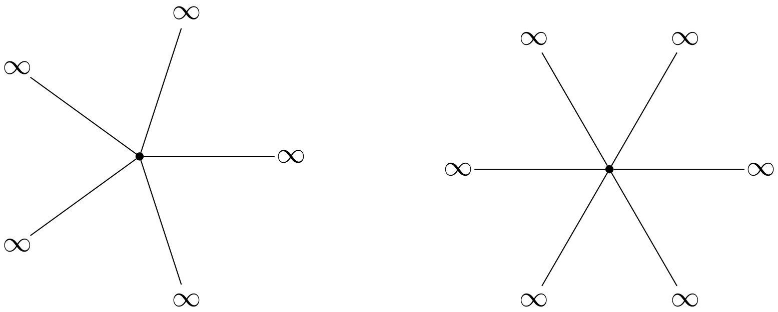
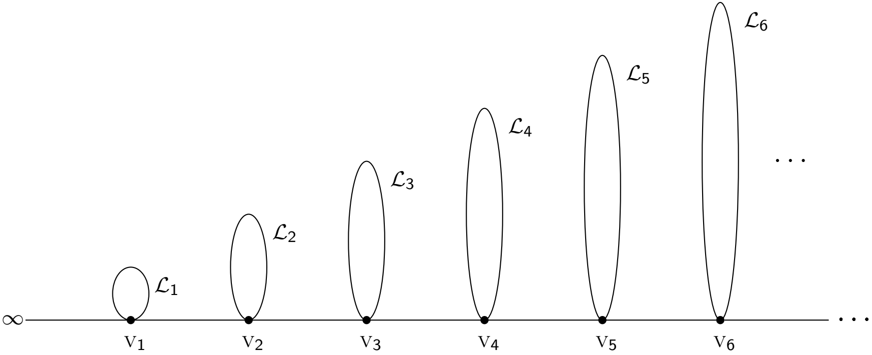


N-star graphs, $N \geq 3$

$$s_\lambda = c_\lambda(\mathcal{G}) < \sigma_\lambda(\mathcal{G}) = \frac{N}{2} s_\lambda$$

Case A2

$c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G})$ and neither infima is attained



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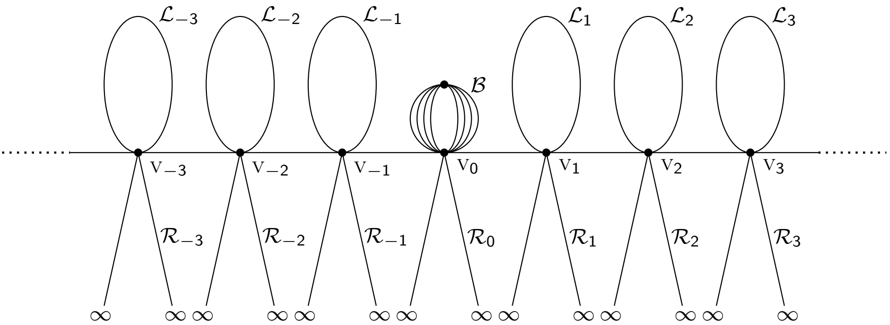

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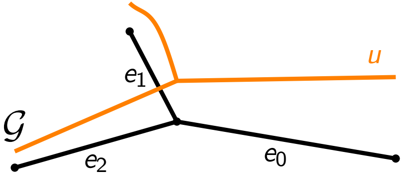
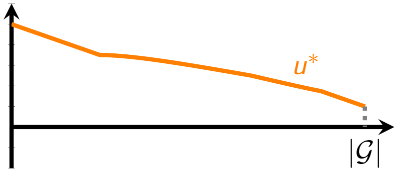
Decreasing rearrangement on the halfline





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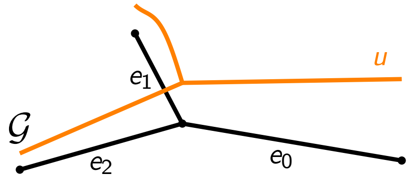

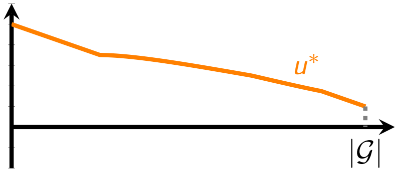

Fundamental property: for all $t > 0$,

$$\mathrm{meas}_{\mathcal{G}}(\{x \in \mathcal{G}, u(x) > t\}) = \mathrm{meas}_{\mathbb{R}^+}(\{x \in]0, |\mathcal{G}|[, u^*(x) > t\}).$$

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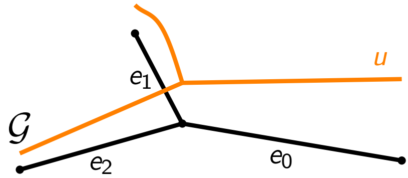

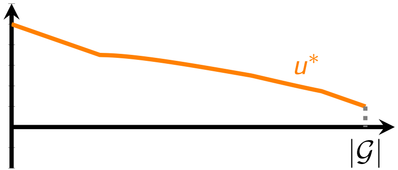

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Consequence: for all $1 \leq p \leq +\infty$,

$$\|u\|_{L^p(\mathcal{G})} = \|u^*\|_{L^p(0,|\mathcal{G}|)}.$$

The Pólya–Szegő inequality

Theorem

Let $u \in H^1(\mathcal{G})$ *be a nonnegative function. Then its decreasing rearrangement* u^* *belongs to* $H^1(0, |\mathcal{G}|)$*, and one has*

$$\|(u^*)'\|_{L^2(0,|\mathcal{G}|)} \leq \|u'\|_{L^2(\mathcal{G})}.$$

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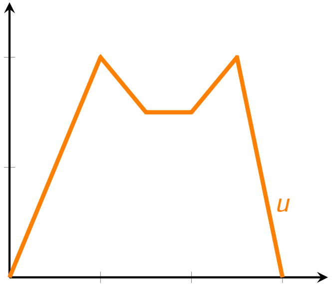
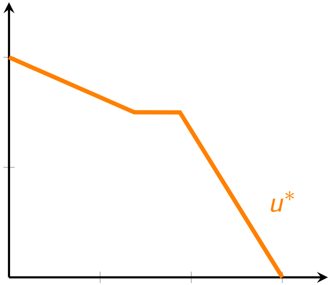
Friedlander, L. *Extremal properties of eigenvalues for a metric graph.* Ann. Inst. Fourier (Grenoble) **55** (2005) no. 1, 199–211.

The Pólya–Szegő inequality

A simple case: affine functions

We assume that u is piecewise affine.





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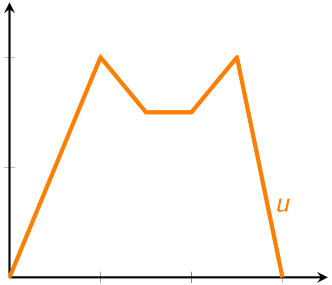

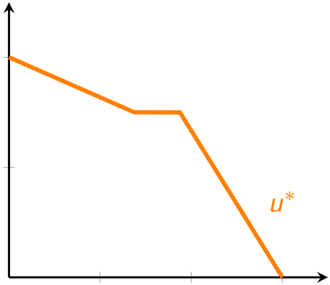

We consider a small open interval $I \subseteq u(\mathcal{G})$ so that $u^{-1}(I)$ consists of a disjoint union of open intervals on which u is affine.

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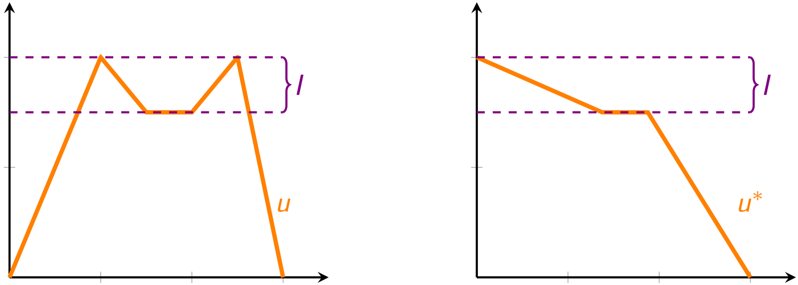

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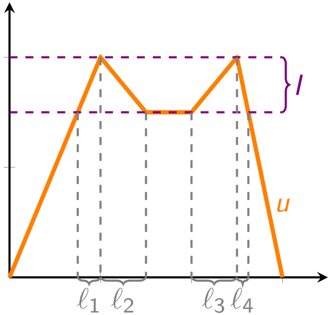

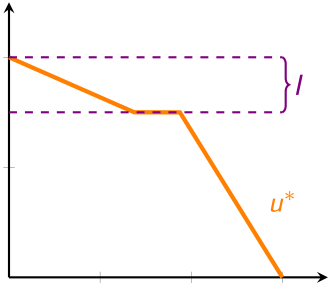

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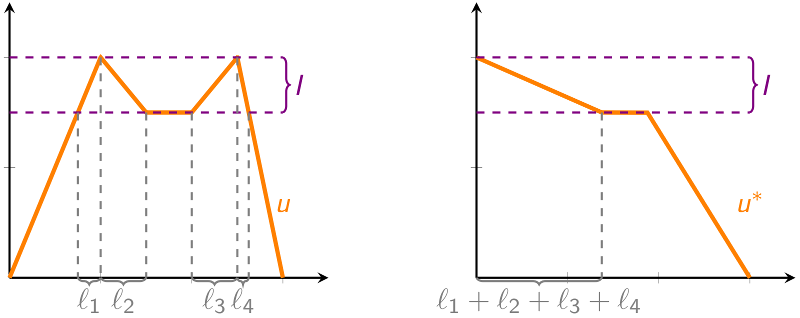

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The Pólya–Szegő inequality

A simple case: affine functions

Original contribution to $\|u'\|_{L^2}^2$:

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A refined Pólya–Szegő inequality...

... or the importance of the number of preimages

Theorem

Let $u \in H^1(\mathcal{G})$ be a nonnegative function. Let $N \geq 1$ be an integer. Assume that, for almost every $t \in]0, \|u\|_\infty[$, one has

$$u^{-1}(\{t\}) = \{x \in \mathcal{G} \mid u(x) = t\} \geq N.$$

Then one has

$$\|(u^*)'\|_{L^2(0,|\mathcal{G}|)} \leq \frac{1}{N}\|u'\|_{L^2(\mathcal{G})}.$$

Assumption (H)

Definition (Adami, Serra, Tilli 2014)

We say that a metric graph $\mathcal{G}$ satisfies assumption (H) if, for every point $x_0 \in \mathcal{G}$, there exist two injective curves $\gamma_1, \gamma_2 : [0, +\infty[\to \mathcal{G}$ parameterized by arclength, with disjoint images except for an at most countable number of points, and such that $\gamma_1(0) = \gamma_2(0) = x_0$.

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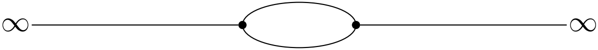

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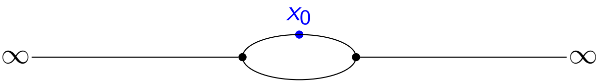

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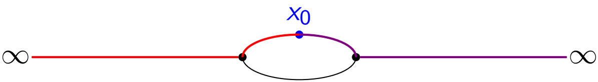

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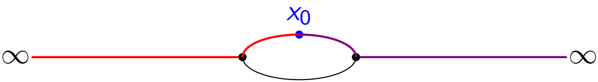

Consequence: *all* nonnegative $H^1(\mathcal{G})$ functions have at least two preimages for almost every $t \in]0, \|u\|_\infty[$.

Non-existence of ground states

Theorem (Adami, Serra, Tilli 2014)

If a metric graph $\mathcal{G}$ has at least one halfline and satisfies assumption (H), then

$$c_\lambda(\mathcal{G}) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u) = s_\lambda$$

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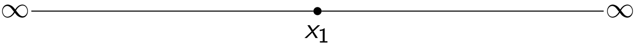
but it is never achieved, unless $\mathcal{G}$ is isometric to one of the exceptional graphs depicted in the next two slides.

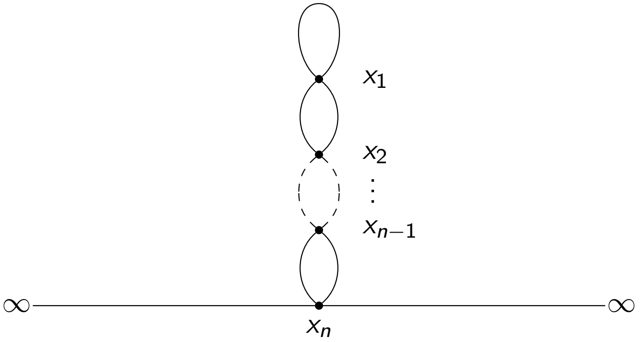
Non-existence of ground states

Exceptional graphs: the real line



Non-existence of ground states

Exceptional graphs: the real line with a tower of circles



A doubly constrained variational problem

We define

$$X_e := \left\{ u \in H^1(\mathcal{G}) \mid \|u\|_{L^\infty(\mathcal{G})} = \|u\|_{L^\infty(e)} \right\}$$

where e is a given bounded edge of $\mathcal{G}$

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Theorem (De Coster, Dovetta, G., Serra (to appear))

*If $\mathcal{G}$ satisfies assumption (H) has a **long enough** bounded edge e, then $c_\lambda(\mathcal{G}, e)$ is attained by a solution $u \in \mathcal{S}_\lambda(\mathcal{G})$, such that $u > 0$ or $u < 0$ on $\mathcal{G}$ and*

$$\|u\|_{L^\infty(e)} > \|u\|_{L^\infty(\mathcal{G} \setminus e)}.$$

Why studying metric graphs?

Mathematical motivations

Main message

Metric graphs allow to study interesting *one dimensional* problems and are much richer then the usual class of intervals of $\mathbb{R}$.

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Replacing $\mathcal{G}$ by noncompact smooth open sets $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ and $H^1(\mathcal{G})$ by $H^1(\Omega)$ or $H^1_0(\Omega)$, one expects that the four cases A1, A2, B1, B2 actually occur.

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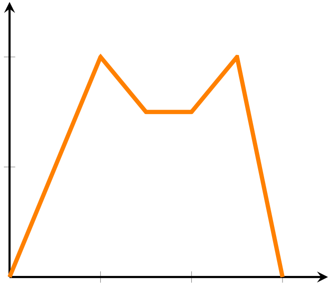
Thanks for your attention!



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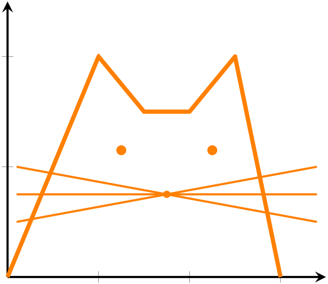

Main papers

- Adami, R., Serra, E., Tilli, P. *NLS ground states on graphs.* Calculus of Variations and Partial Differential Equations, 54(1), 743-761 (2015).
- De Coster C., Dovetta S., Galant D., Serra E. *On the notion of ground state for nonlinear Schrödinger equations on metric graphs.* To appear.

Overviews of the subject

Adami R. *Ground states of the Nonlinear Schrodinger Equation on Graphs: an overview (Lisbon WADE).* https://www.youtube.com/watch?v=G-FcnRVvoos (2020)

Adami R., Serra E., Tilli P. *Nonlinear dynamics on branched structures and networks.* https://arxiv.org/abs/1705.00529 (2017)

Kairzhan A., Noja D., Pelinovsky D. *Standing waves on quantum graphs.* J. Phys. A: Math. Theor. 55 243001 (2022)

An application: atomtronics

- A *boson*[2] is a particle with integer spin.

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- Since 2000: emergence of *atomtronics*, which studies circuits guiding the propagation of ultracold atoms.

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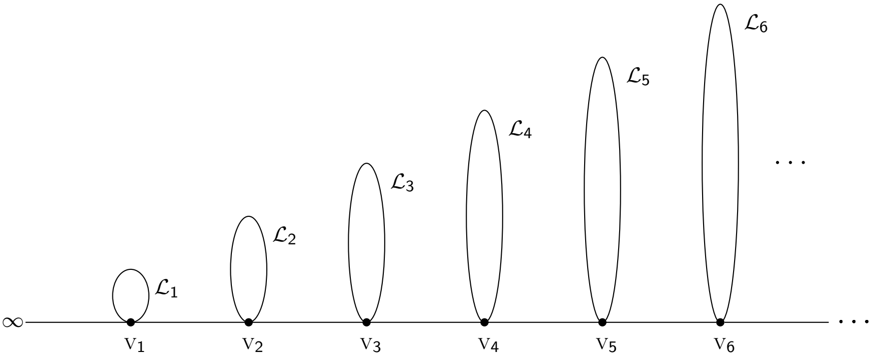
What's going on in case A2?

$c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G})$ and neither infima is attained



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- One obtains

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so

$$c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G}) = s_\lambda$$

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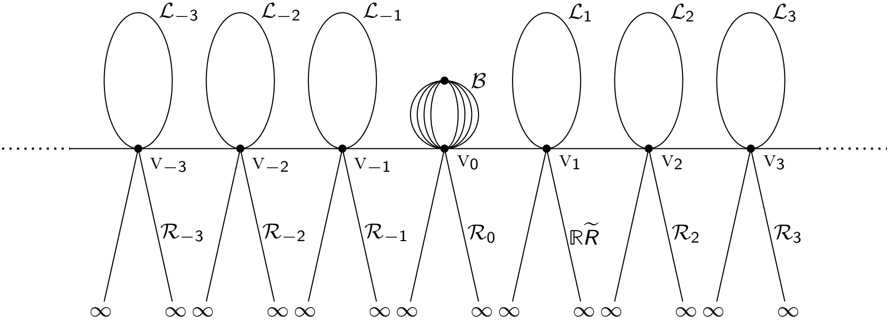
What's going on in case B2?

$c_\lambda(\mathcal{G}) < \sigma_\lambda(\mathcal{G})$ and neither infima is attained

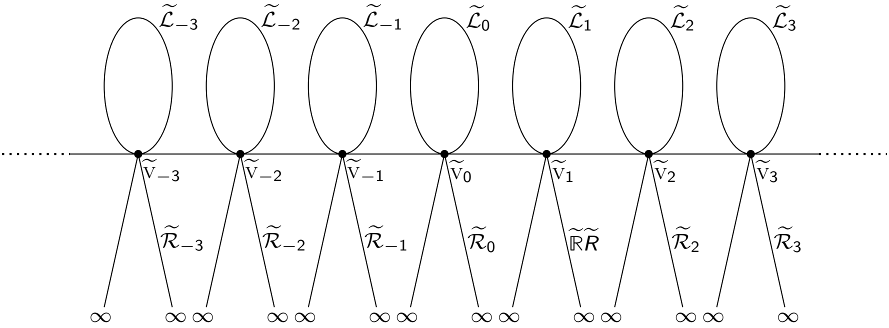


The graph $\mathcal{G}_N$.

The loops $\mathcal{L}_i$ have length N and $\mathcal{B}$ is made of N edges of length 1.

What's going on in case B2?

A second, periodic, graph



The graph $\widetilde{\mathcal{G}}_N$.

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Two problems at infinity

- Since $\mathcal{G}_N$ and $\widetilde{\mathcal{G}}_N$ satisfy (H) and contain halflines, one has

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  but that $\sigma_\lambda(\mathcal{G}_N)$ *is not attained*.
- Therefore, for large N, we have that

$$s_\lambda = c_\lambda(\mathcal{G}_N) < \sigma_\lambda(\mathcal{G}_N),$$

  and neither infima is attained, as claimed.